

Urysohn's metrization theorem for higher cardinals

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Abstract

In this paper a generalization of Urysohn's metrization theorem is given for higher cardinals. Namely, it is shown that a topological space with a basis of cardinality at most $|\omega_\mu|$ or smaller is ω_μ -metrizable if and only if it is ω_μ -additive and regular, or, equivalently, ω_μ -additive, zero-dimensional, and T_0 . Furthermore, all such spaces are shown to be embeddable in a suitable generalization of Hilbert's cube.

Keywords: ω_μ -metric space, Urysohn's metrization theorem, embedding theorem, ω_μ -additive space

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1. Introduction

In this section the concept of ω_μ -metric spaces is defined and briefly discussed. For a more elaborate description of ω_μ -metric spaces, see e.g. [1]. Section 2 is dedicated to some preliminary results which are then used to prove an extension of Urysohn's metrization theorem in section 3.

A number of properties of a topological space equivalent with ω_μ -metrizable were given by Hodel [2] and by Nyikos and Reichel [1]. The special case of spaces with a basis of cardinality at most $|\omega_\mu|$ seems not to have been considered. A complete characterization of such ω_μ -metrizable topological spaces is exhibited here in Theorem 2 in terms of simple topological properties which may be easier to verify than those required by more general metrization theorems.

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If G is an ordered abelian group, X is a nonempty set and $d: X \times X \rightarrow G$ is a function such that

1. $d(x, y) \geq 0$ for all $x, y \in X$,
2. $d(x, y) = 0$ only if $x = y$,
3. $d(x, y) = d(y, x)$ for all $x, y \in X$, and
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$,

then d is a *metric of X over G* . Any such metric d gives rise to a topology of X whose basis consists of the open balls

$$B_d(x, r) = \{y \in X : d(x, y) < r\}, \quad (1)$$

for all $x \in X$ and $r \in G$, $r > 0$. Where no confusion is possible, the subscript d is omitted. If X is a topological space and there is a metric d of X over G giving rise to the same topology, then X is *G -metrizable* and the pair (X, d) is a *G -metric space*.

Of special interest are the groups \mathbb{Z}^α and \mathbb{R}^α , where α is any non-zero ordinal, with lexicographical order and componentwise addition. Any element $x \in \mathbb{Z}^\alpha$ is a sequence $(x_\lambda)_{\lambda \in \alpha}$ indexed by α with each $x_\lambda \in \mathbb{Z}$, and similarly for \mathbb{R}^α .

For any $\lambda \in \alpha$, we define elements r^λ in \mathbb{Z}^α (or in \mathbb{R}^α) by setting $r^\lambda_\lambda = 1$ and $r^\lambda_\nu = 0$ for all $\nu \neq \lambda$. One can immediately see that if α is a regular ordinal and (X, d) is a \mathbb{Z}^α -metric (or \mathbb{R}^α -metric) space, then the collection

$$\{B_d(x, r^\lambda) : x \in X, \lambda \in \alpha\} \quad (2)$$

is a basis for the topology of X .

Proposition 1. *If a topological space X is \mathbb{Z}^α -metrizable, it is also $\mathbb{Z}^{\text{cf } \alpha}$ -metrizable, where $\text{cf } \alpha$ is the cofinality of α . Similarly, if X is \mathbb{R}^α -metrizable, it is also $\mathbb{R}^{\text{cf } \alpha}$ -metrizable.*

Proof. Let X be a \mathbb{Z}^α -metric space, and let $L \subset \alpha$ be a cofinal subset order isomorphic to $\text{cf } \alpha$. Then the topology given by the basis

$$\{B(x, r^\lambda) : x \in X, \lambda \in L\} \quad (3)$$

is immediately seen to be identical to the original metric topology of X . Thus X is \mathbb{Z}^L -metrizable. The same argument holds for \mathbb{R}^α -metrizable spaces. \square

Proposition 2. *Let α be an infinite regular ordinal. A topological space X is \mathbb{Z}^α -metrizable if and only if it is \mathbb{R}^α -metrizable.*

Proof. Since \mathbb{Z}^α is a subgroup of \mathbb{R}^α , any \mathbb{Z}^α -metric space is trivially an \mathbb{R}^α -metric space.

Let then (X, d) be a \mathbb{R}^α -metric space. For any points $x, y \in X$, $x \neq y$, let $n_{xy} = \min\{\lambda \in \alpha : d_\lambda(x, y) \neq 0\}$. One can define a \mathbb{Z}^α -metric δ for X by setting $\delta(x, x) = 0$ and $\delta(x, y) = r^{n_{xy}}$ when $x \neq y$.

Let $x, y, z \in X$ be any three distinct points. Because d obeys the triangle inequality, one has $\min\{n_{xz}, n_{yz}\} \leq n_{xy}$, and so $\max\{r^{n_{xz}}, r^{n_{yz}}\} \geq r^{n_{xy}}$. This implies $\max\{\delta(x, z), \delta(y, z)\} \geq \delta(x, y)$, and hence $\delta(x, z) + \delta(y, z) \geq \delta(x, y)$. Thus δ obeys the triangle inequality; the other conditions for a metric are obviously fulfilled. Since $B_\delta(x, r^\lambda) \supset B_d(x, r^{\lambda+1})$ and $B_d(x, r^\lambda) \supset B_\delta(x, r^{\lambda+1})$ for all $x \in X$ and $\lambda \in \alpha$, the two metric topologies are the same. \square

Due to Proposition 1 only regular ordinals α need to be considered. Every infinite regular ordinal α is an initial ordinal, that is $\alpha = \omega_\mu$ for some ordinal μ . Moreover, for a finite α one has $\text{cf } \alpha = 1$, which yields either the discrete \mathbb{Z}^1 -metric or the usual \mathbb{R}^1 -metric.

Definition 1. A topological space X is an ω_μ -metric space if it is a \mathbb{Z}^{ω_μ} -metric space (equivalently \mathbb{R}^{ω_μ} -metric space by Proposition 2).

Unless otherwise stipulated, every ω_μ -metric will be assumed to take values in \mathbb{Z}^{ω_μ} instead of \mathbb{R}^{ω_μ} .

2. Preliminaries

Let κ be a cardinal. A topological space X is κ -additive if for any collection $\{U_i : i \in I\}$ of open sets in X the intersection $\bigcap_{i \in I} U_i$ is open whenever $|I| < \kappa$. An $|\omega_\mu|$ -additive space is also called an ω_μ -additive space.

Proposition 3. *Let X be topological space with a basis of cardinality κ or smaller. Then*

1. *X contains a dense set whose cardinality is at most κ , and*
2. *every open cover of X has a subcover whose cardinality is at most κ .*

Proof. Let \mathcal{B} be a basis for the topology of X such that $|\mathcal{B}| \leq \kappa$.

1. For every set $A \in \mathcal{B}$ there is an element $x_A \in A$. The set $\{x_A : A \in \mathcal{B}\}$ is obviously dense and its cardinality cannot exceed κ .

2. Let \mathcal{C} be any open cover of X . If \mathcal{B}' is the collection of sets $B \in \mathcal{B}$ such that $B \subset U$ for some $U \in \mathcal{C}$, one can choose for every $B \in \mathcal{B}'$ a set U_B with $B \subset U_B \in \mathcal{C}$. The collection $\mathcal{C}' = \{U_B : B \in \mathcal{B}'\}$ is the subcover sought for: indeed, each $x \in X$ is an interior point of some $U \in \mathcal{C}$, and so $x \in B \subset U$ for some $B \in \mathcal{B}$ and thus $\bigcup_{B \in \mathcal{B}'} U_B = X$. \square

Proposition 4. *In any ω_μ -metric space X either one of the following two conditions is sufficient to guarantee that the topology of X have a basis of cardinality $|\omega_\mu|$ or smaller:*

1. *X contains a dense subset whose cardinality is at most $|\omega_\mu|$.*
2. *Every open cover of X has a subcover whose cardinality is at most $|\omega_\mu|$.*

Proof. 1. Let $A \subset X$ be a dense subset such that $|A| \leq |\omega_\mu|$. The collection $\mathcal{B} = \{B(a, r^\lambda) : a \in A, \lambda \in \omega_\mu\}$ has obviously cardinality $|\omega_\mu|$ or smaller. It remains to show that \mathcal{B} is also a basis for the topology. Let $U \subset X$ be any nonempty open set and let $x \in U$. Then there is $\lambda \in \omega_\mu$ such that $B(x, r^\lambda) \subset U$, and one can find a point $a \in A \cap B(x, r^{\lambda+1})$. Now $x \in B(a, r^{\lambda+1}) \subset B(x, r^\lambda) \subset U$ and $B(a, r^{\lambda+1}) \in \mathcal{B}$. Hence \mathcal{B} indeed is a basis.

2. For every $\lambda \in \omega_\mu$ the collection $\{B(x, r^\lambda) : x \in X\}$ is an open cover of X . Therefore there is a set $A_\lambda \subset X$ such that $|A_\lambda| \leq |\omega_\mu|$ and $\{B(x, r^\lambda) : x \in A_\lambda\}$ is an open cover of X . The union $A = \bigcup_{\lambda \in \omega_\mu} A_\lambda$ is dense in X and has cardinality $|A| \leq |\omega_\mu| \times |\omega_\mu| = |\omega_\mu|$. \square

Lemma 1. *Let X be a T_3 -space and let κ be a cardinal. Assume that X is κ -additive and every open cover of X has a subcover of cardinality κ or smaller. Then X is a T_4 -space.*

Proof. Let E and F be disjoint nonempty closed sets in X . Since X is T_3 , for every $e \in E$ one can find a neighborhood U_e such that $F \subset X \setminus \overline{U_e}$. Similarly every $f \in F$ has a neighborhood V_f with $E \subset X \setminus \overline{V_f}$. Since

$$\mathcal{C} = \{X \setminus (E \cup F)\} \cup \{U_e : e \in E\} \cup \{V_f : f \in F\} \quad (4)$$

is an open cover of X , the assumed covering property guarantees that \mathcal{C} has a subcover

$$\mathcal{C}' = \{X \setminus (E \cup F)\} \cup \{U_{e_\lambda} : \lambda \in \alpha\} \cup \{V_{f_\lambda} : \lambda \in \alpha\}, \quad (5)$$

where each $e_\lambda \in E$ and $f_\lambda \in F$, and where α is the initial ordinal of the cardinal κ .

For every $\lambda \in \alpha$ the sets

$$\begin{aligned} A_\lambda &= U_{e_\lambda} \setminus \bigcup_{\nu < \lambda} \overline{V}_{f_\nu} \quad \text{and} \\ B_\lambda &= V_{f_\lambda} \setminus \bigcup_{\nu < \lambda} \overline{U}_{e_\nu} \end{aligned} \tag{6}$$

are open by hypothesis, and hence the sets

$$A = \bigcup_{\lambda \in \alpha} A_\lambda \quad \text{and} \quad B = \bigcup_{\lambda \in \alpha} B_\lambda \tag{7}$$

are neighborhoods of E and F , respectively. These neighborhoods are disjoint; for if $\nu \leq \lambda$, then $\overline{B}_\nu \subset X \setminus A_\lambda$, and so $A_\lambda \cap B_\nu = \emptyset$ and similarly in the case $\nu \geq \lambda$. \square

The following two known lemmas are elementary, and they are included only for the sake of an easy reference.

Lemma 2. *If a topological space X is zero-dimensional and T_0 , then it is also T_2 and T_3 .*

Lemma 3. *Let X be a topological T_3 -space with a basis \mathcal{B} and let $x \in X$. Then for each neighborhood U of x there are $B, B' \in \mathcal{B}$ such that $x \in B \subset \overline{B} \subset B' \subset U$.*

3. The metrization theorem

In order to extend Urysohn's metrization theorem to higher cardinals and thus to ω_μ -metric spaces, a generalization of Hilbert's cube is needed. The product topology of $\{0, 1\}^{\omega_\mu}$ is not suitable for this purpose, since it is not ω_μ -additive for $\mu > 0$.

Definition 2. Let \mathbb{Z}^{ω_μ} be given an ω_μ -metric d by defining $d(x, y)_\lambda = |x_\lambda - y_\lambda|$ for every $\lambda \in \omega_\mu$. The set $Q_\mu = \{0, 1\}^{\omega_\mu} \subset \mathbb{Z}^{\omega_\mu}$ with the ω_μ -metric inherited from \mathbb{Z}^{ω_μ} is the *generalized Hilbert's cube*.

A basis for the topology of the cube Q_μ consists of the products $\prod_{\lambda \in \omega_\mu} U_\lambda$, where there is $\nu \in \omega_\mu$ such that U_λ is a singleton when $\lambda < \nu$ and $U_\lambda = \{0, 1\}$ when $\lambda \geq \nu$. The cardinality of this basis is $|\omega_\mu|$.

The embedding theorem 1, which will be stated and proven shortly, will make use of the classical Urysohn's lemma:

Lemma 4 (Urysohn's lemma). *Let X be a T_4 -space and let E and F be disjoint nonempty closed sets in X . Then there is a continuous mapping $f: X \rightarrow [0, 1]$ which satisfies $f(E) = \{0\}$ and $f(F) = \{1\}$.*

Lemma 5. *Let X be a T_4 -space and let E and F be disjoint nonempty closed sets in X . If X is ω_1 -additive, there is a continuous mapping $f: X \rightarrow \{0, 1\}$ which satisfies $f(E) = \{0\}$ and $f(F) = \{1\}$.*

Proof. Let $g: X \rightarrow [0, 1]$ be a mapping given by Urysohn's lemma. Define $f: X \rightarrow \{0, 1\}$ by setting $f(x) = 0$ when $g(x) = 0$ and $f(x) = 1$ otherwise. Every set $g^{-1}([0, 1/n))$, $n \in \mathbb{N}$, is open in X and therefore, by hypothesis, so is the intersection $f^{-1}(\{0\}) = g^{-1}(\{0\}) = \bigcap_{n \in \mathbb{N}} g^{-1}([0, 1/n))$. Thus f is continuous. \square

Theorem 1. *Let X be a topological T_1 - and T_3 -space. If X is ω_μ -additive and has a basis of cardinality $|\omega_\mu|$ or smaller for a regular ordinal ω_μ , $\mu > 0$, then X can be embedded in the generalized Hilbert's cube Q_μ .*

Proof. It follows from Proposition 3 and Lemma 1 that X is also T_4 .

Let $\{B_j \subset X : j \in J\}$ be a basis for X such that $|J| \leq |\omega_\mu|$. Let P be the set of pairs $(i, j) \in J \times J$ for which $\overline{B_i} \subset B_j$. Since $|P| \leq |J|$, the elements of P can be indexed so that $P = \{(i_\lambda, j_\lambda) : \lambda \in \omega_\mu\}$.

For every $\lambda \in \omega_\mu$ a continuous mapping $f_\lambda: X \rightarrow \{0, 1\}$ is chosen such that $f_\lambda(\overline{B_{i_\lambda}}) = \{1\}$ and $f_\lambda(X \setminus B_{j_\lambda}) = \{0\}$. This is possible by Lemma 5. We define $f: X \rightarrow Q_\mu = \{0, 1\}^{\omega_\mu}$ componentwise by the mappings f_λ and show that f embeds X in Q_μ .

The mapping f is continuous: Let $x \in X$ be a point and let U be a neighborhood of $f(x)$. Then there exist $\nu \in \omega_\mu$ and sets $U_\lambda \subset \{0, 1\}$ with $f_\lambda(x) \in U_\lambda$ for all $\lambda \in \omega_\mu$ and $U_\lambda = \{0, 1\}$ when $\lambda \geq \nu$, so that U contains the product $\prod_{\lambda \in \omega_\mu} U_\lambda$. Because each f_λ is continuous, for every $\lambda < \nu$ one can find an open set $V_\lambda \subset X$ so that $f_\lambda(V_\lambda) \subset U_\lambda$. Since X is ω_μ -additive, the set $V = \bigcap_{\lambda < \nu} V_\lambda$ is a neighborhood of x . Obviously $f(V) \subset \prod_{\lambda \in \omega_\mu} U_\lambda \subset U$.

The mapping f is one-to-one: Let $x, y \in X$ be two distinct points. By the T_1 -property and Lemma 3 there are $i, j \in J$ such that $x \in B_i \subset \overline{B_i} \subset B_j$. Thus $(i, j) = (i_\lambda, j_\lambda)$ for some $\lambda \in \omega_\mu$. Now $f_\lambda(x) = 1$ and $f_\lambda(y) = 0$, and so $f(x) \neq f(y)$.

Let $g: f(X) \rightarrow X$ be the inverse of f . It remains to show that g is continuous. Fix $x \in X$ and let U be a neighborhood of x . By Lemma 3 there is $\lambda \in \omega_\mu$ for which $x \in B_{i_\lambda}$ and $B_{j_\lambda} \subset U$. Now $f_\lambda(x) = 1$ and the set

$V = \prod_{\nu \in \omega_\mu} V_\nu$, where $V_\lambda = \{1\}$ and $V_\nu = \{0, 1\}$ for $\nu \neq \lambda$, is a neighborhood of $f(x)$ in Q_μ . For every $y \in g(V \cap f(X))$ we have $f(y) \in V$, $f_\lambda(y) = 1$, and so $y \in B_{j_\lambda} \subset U$. Thus $g(V \cap f(X)) \subset U$ and g is continuous. \square

Theorem 2. *For any topological space X and any regular ordinal $\omega_\mu > \omega_0$ the following are equivalent:*

1. X is ω_μ -additive, T_0 , zero-dimensional and has a basis of cardinality $|\omega_\mu|$ or smaller.
2. X is ω_μ -additive, T_1 and T_3 , and has a basis of cardinality $|\omega_\mu|$ or smaller.
3. X is ω_μ -metrizable and has a basis of cardinality $|\omega_\mu|$ or smaller.
4. X is ω_μ -metrizable and contains a dense set of cardinality $|\omega_\mu|$ or smaller.
5. X is ω_μ -metrizable and every open cover of X has a subcover of cardinality $|\omega_\mu|$ or smaller.
6. X can be embedded in Q_μ .

Proof. Lemma 2 and Theorem 1 provide the implications $1 \Rightarrow 2$ and $2 \Rightarrow 6$. The generalized Hilbert's cube Q_μ is ω_μ -metrizable by definition and has a basis of cardinality $|\omega_\mu|$, whence $6 \Rightarrow 3$. By Propositions 3 and 4 the conditions 3, 4, and 5 are equivalent.

The implication $4 \Rightarrow 1$ is seen as follows. Let A be a dense subset of X such that $|A| \leq |\omega_\mu|$. The collection $\{B(a, r^\lambda) : a \in A, \lambda \in \omega_\mu\}$ can easily be verified to be a clopen basis for X , and its cardinality is manifestly $|\omega_\mu|$ or smaller. \square

Remark 1. The set Q_μ is considered to be a generalization of Hilbert's cube due to its role in Theorem 2. The cube Q_0 , however, is a Cantor set with its usual topology, and so the theorem does not hold for $\mu = 0$.

Remark 2. Let ω_μ , $\mu > 0$, be a regular ordinal. Consider a topological space X which has two bases, one of cardinality $|\omega_\mu|$ or smaller and the other consisting of clopen sets. Does X have a basis which has both of these properties?

First, X can be assumed to be a T_0 -space; it is sufficient to consider the bases for the Kolmogorov quotient $KQ(X)$ of X , and $KQ(X)$ is a T_0 -space. By Theorem 2 ω_μ -additivity is sufficient for such a basis to exist. If X is strongly zero-dimensional, every open cover of X has a refinement where the covering sets are disjoint. Any such refinement of the basis for X that has

cardinality $|\omega_\mu|$ or smaller is a suitable clopen basis. Without any further assumptions, however, it is unclear whether or not such a basis exists.

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